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Elementary Waves for Hyperbolic Equations in Higher Space Dimensions: An Example from Petroleum Reservoir Modeling

J. Glimm and D. H. Sharp

Research and Development Report

Supported by the Applied Mathematical Sciences
subprogram of the Office of Energy Research,
U.S. Department of Energy under
Contract DE-AC02-76ER03077

Mathematics and Computers

May 1985

NEW YORK UNIVERSITY

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Printed in U.S.A.

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Springfield, VA 22161

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ELEMENTARY WAVES FOR HYPERBOLIC EQUATIONS IN HIGHER SPACE DIMENSIONS: AN EXAMPLE FROM PETROLEUM RESERVOIR MODELING

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ABSTRACT

Elementary waves are defined as scale invariant solutions of a nonlinear hyperbolic conservation law which are stationary in a suitable frame. We show by example the importance of next to leading order scaling data, which breaks the scale invariance for these elementary waves. The example is drawn from petroleum reservoir modeling and is of interest in its own right.

The crossing of an oil-water bank with a geological discontinuity is not a simple jump discontinuity, and defines a discontinuity of co-dimension two. The dynamically stable forms of these co-dimension two discontinuities define an elementary wave of co-dimension two. In this note, we determine the possible forms for this elementary wave. The solution which appears to be generic is not scale invariant.

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[1] Supported in part by the National Science Foundation, grant DMS - 831229.

[2] Supported in part by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U. S. Department of Energy, under contract DE-AC02-76ER03077

[3] Supported in part by the Army Research Office, grant DAAG29-83-K-0007

[4] Work supported by the Department of Energy.

1. Elementary Waves

Elementary waves and Riemann solutions are solutions of a nonlinear system of conservation laws

$$u_t + \nabla \cdot f(u) = 0 \tag{1.1}$$

which are distinguished by symmetry properties. We take (1.1) to be an $n \times n$ system, considered in d space variables. A Riemann solution is a solution u of (1.1) which is invariant under the scale transformation $\vec{x}, t \to s\vec{x}, st$, $s > 0$. An elementary wave with velocity $\vec{\sigma} \in R^d$ is a Riemann solution which is invariant under the group action $\vec{x}, t \to \vec{x} + \vec{\sigma}\tau, t + \tau$. Equivalently these solutions are transformed into a stationary solution ($u_t = 0$) by a Galilean transformation $\vec{x}, t \to \xi, t = \vec{x} - \vec{\sigma}t, t$. The elementary waves and Riemann solutions are said to have co-dimension j if they are independent of all but j of their space coordinates. For a more extensive discussion of these concepts, see [1,2].

In this paper we show by example the importance of next to leading order scaling data, which break the scale invariance for these elementary waves. The scale symmetry breaking solution contains a length scale. The example is important in its own right, because the geological layers are an important factor in the modeling of petroleum reserviors.

2. An Example from Petroleum Reservoir Modeling

The equations for flow in porous media are

$$v = - \lambda(s) \nabla p \tag{2.1}$$

$$\nabla \cdot v = 0 \tag{2.2}$$

$$s_t + v \cdot \nabla f(s) = 0 \, , \tag{2.3}$$

with the neglect of capillary diffusion in the saturation equation (2.3). Equation (2.3) is a scalar hyperbolic equation which admits discontinuities, known as oil or water banks. We consider a vertical cross section of a three dimensional reservoir. In this case $\lambda = \lambda(s,x,y)$ has prominent discontinuities across the boundaries of distinct geological layers. The meeting point of an oil bank and a geological layer is a discontinuity for (2.1) - (2.3) of co-dimension two. Since it should move with a definite velocity (along the fixed layer boundary) and with a constant geometrical form, it defines an elementary wave of co-dimension two.

The purpose of this note is to find the possible configurations and velocities of this elementary wave. Because of the assumed scale invariance of the solution, all data is given with lowest order scaling behavior only, and is thus scale invariant also. The discontinuities divide the plane into four sectors, bounded by rays, while the saturations and velocities are constant in each such sector. The elementary wave is then defined by these saturations and velocities together with the angles between the rays.

The equations for a co-dimension two elementary wave are of two types. There are jump conditions across each discontinuity and there is a dynamical consistency equation, stating that in some Galilean frame, the solution is stationary.

We find three solutions to these equations. One solution has all flow parallel to the oil water bank, and one has all flow normal to the oil water bank. Very specific angles of the oil water bank are required for these solutions. The third elementary wave has data given to one degree higher order. It violates scale invariance, having next to leading order scaling data. The oil water bank is tangent to the layer at the crossing point, and curves away on either side. This elementary wave appears to be the generic solution.

3. The Jump Equations

Let us take the layer located on the x axis, with the region $y < 0$ having absolute permeability $M \neq 1$ and the region $y > 0$ having absolute permeability 1. We make the simplifying assumption that the fractional flow function f in (2.3) is the same on both sides of the layer discontinuity. This is correct only in so far as f depends on the fluid properties (the oil and water) but does not allow for a change in the influence of the rock properties on f. With this assumption, we can suppose that the saturation values are chosen to be constant in each of the four sectors, and at a value such that the wave connecting them in (2.3) is a shock. Also the frontal mobility ratio M_o is the same in both layers, under this assumption. We follow the terminology defined by Fig. 1, so that n and t are the unit normal and tangent vectors to the bank in the upper layer and $\tilde{n}$ and $\tilde{t}$ are the same in the lower layer. Also we can suppose that the velocities are constant in each sector, as defined by Fig. 1. The jump equations for the velocities in each sector are

$$< n, v - v' > = 0 = < t, v - M_o v' > \tag{3.1}$$

$$< \tilde{n}, \tilde{v} - \tilde{v}' > = 0 = < \tilde{t}, \tilde{v} - M_o \tilde{v}' > \tag{3.2}$$

The jump equations for the layer discontinuity are

$$\tilde{v}_y = v_y \, , \quad \tilde{v}_y' = v_y' \tag{3.3}$$

$$\tilde{v}_x = M v_x \, , \quad \tilde{v}_x' = M v_x' \tag{3.4}$$

From (3.1) we have $v - v'$ parallel to t and $v - M_o v'$ parallel to n and so

$$< v - v', v - M_o v' > = 0 \, . \tag{3.5}$$

Similarly

$$< \tilde{v} - \tilde{v}', \tilde{v} - M_o \tilde{v}' > = 0 \, . \tag{3.6}$$

Let $\mathcal{M}$ denote the 2 × 2 matrix

$$\mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} .$$

Then $\tilde{v} = \mathcal{M} v$ and $\tilde{v}' = \mathcal{M} v'$ so that

$$< v - v', \mathcal{M}^2 (v - M_o v') > = 0 \, . \tag{3.7}$$

Taking linear combinations of (3.5) and (3.7) yields the following form of the jump relations:

$$(v_x - v_x')\,(v_x - M_o v_x') = 0. \tag{3.8a}$$

$$(v_y - v_y')\,(v_y - M_o v_y') = 0. \tag{3.8b}$$

There are four solutions to (3.3), two of which satisfy the dynamical consistency equations and thus define elementary waves.

4. The Tangential Flow and the Normal Flow Elementary Waves

4.1. $M_o v' = v \; , \; v \neq 0$

From (3.1), $v - v'$ is parallel to the bank. Since v and v' are parallel to each other, v and v' are parallel to the bank. In this case the elementary wave is stationary and the dynamical consistency equation is satisfied.

4.2. $v' = v \; , \; v \neq 0$

From (3.1), $v - M_o v'$ is normal to the bank, and thus so are v and v'. Let

$$v_n = < v, n > , \quad \tilde{v}_n = < \tilde{v}, \tilde{n} > .$$

Then the velocity of the crossing point is $v_n / \sin \theta$, and the dynamical consistency equation is

$$v_n / \sin \theta = \tilde{v}_n / \sin \tilde{\theta} \; ,$$

or

$$v_n \sin \tilde{\theta} = \tilde{v}_n \sin \theta \; . \tag{4.1}$$

Because v is normal to the bank, we have

$$n = \frac{v}{||v||} \; , \quad |v_n| = ||v|| \; ,$$

$$\sin \theta = n_x = \frac{v_x}{||v||}$$

and analogous $\tilde{v}$ relations. Substitution in (4.1) yields

$$||v|| \, \frac{\tilde{v}_x}{||\tilde{v}||} = ||\tilde{v}|| \, \frac{v_x}{||v||}$$

or

$$M \, ||v||^2 = ||\tilde{v}||^2$$

since $\tilde{v}_x = M\, v_x$. Thus

$$M\,(v_x^2 + v_y^2) = M^2 v_x^2 + v_y^2$$

which yields the equation

$$v_y = \pm M^{\frac{1}{2}} v_x \, .$$

4.3. $v_y = v_y'$, $v_x = M_o v_x'$, $v \neq 0$.

In this case $v - v' \neq 0$, so $v - v'$ is parallel to t and consequently $t = e_x$, $n = -e_y$ where e_x and e_y are the unit basis vectors in R^2. The bank and the layer are parallel. The bank moves away from the layer dynamically, so this case does not define an elementary wave.

4.4. $v_x = v_x'$, $v_y = M_o v_y'$, $v \neq 0$.

In this case t is parallel to $v - v'$ and consequently $t = e_y$, $n = e_x$. The bank and layer are normal. This case is not dynamically stable and does not define an elementary wave.

5. The Elementary Wave which Breaks Scale Invariance

We have found two solutions for the elementary wave equations which have $v \neq 0$. All other solutions must have $v = 0$ at the crossing point. We assume $v = 0$ at the crossing point and expand v in a Taylor's series about the crossing point,

$$v = x\, v_x + y\, v_y = x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y}$$

and similarly for v' and the velocities in the other sectors. Also we suppose that the normal to the bank is taken to first order and thus the bank is represented as an arc of a circle, near the layer boundary.

With v and the normals taken to first order near the cross point, consider the jump and consistency equations to the same order. In general a solution of these equations will lead to a dynamical evolution of the crossing angle and a fixed crossing position. Such a solution is not (by definition) an elementary wave. There is one exception, however, which occurs when the bank crosses the layer boundary tangentially (with first order contact). We examine this case in more detail.

To first order, the velocity along the layer is

$$v = x\, v_{,x} = x \frac{\partial v}{\partial x} ,$$

with similar expressions for v', $\tilde{v}$, and $\tilde{v}'$. The jump conditions can be analysed as before and have four solutions. Of these, only the third of the above solutions is compatible with the zeroth order data, namely $n = -e_y$, $t = e_x$. Thus

$$\frac{\partial v_x}{\partial x} = M_o \frac{\partial v'_x}{\partial x}, \text{ and } \frac{\partial v_y}{\partial x} = \frac{\partial v'_y}{\partial x} , \quad \frac{\partial v}{\partial x} \neq 0 .$$

The crossing point velocity is determined by the radius of curvature of the bank. Thus dynamical consistency is a condition on $\frac{r}{\tilde{r}}$, where r and $\tilde{r}$ are the radii of curvature above and below the bank respectively.

In the notation defined by Figure 2, the dynamical consistency equation is $dx = d\tilde{x}$. Here we write

$$dy = v_y\, dt = dx\, v_{y,x}\, dt .$$

But $dx = r d\phi$ up to higher order terms and

$$dy = r(1 - \cos d\phi) = r \frac{(d\phi)^2}{2} .$$

Equating these expressions for dy, we obtain

$$v_{y,x}\, dt = \frac{1}{2}\, d\phi$$

and so

$$dx = 2r\, v_{y,x}\, dt .$$

Because $n = -e_y$, we have

$$v_{y,x} = v'_{y,x} = \tilde{v}_{y,x} = \tilde{v}'_{y,x} ,$$

and so $dx = d\tilde{x}$ implies $r = \tilde{r}$. This is the dynamical consistency equation. The equation

$$\frac{dx}{dt} = 2r\, v_{y,x} \; .$$

defines the velocity of the crossing point.

6. References

1. J. Glimm, C. Klingenberg, O. Mcbryan, B. Plohr, D. Sharp and S. Yaniv. Front Tracking and Two Dimensional Riemann Problems. Adv. Appl. Math., to appear.
2. J. Glimm and D. Sharp. An S Matrix Theory for Classical Nonlinear Physics. Foundations of Physics. To appear.

7. Figure Captions

Figure 1. A bank crossing a layer discontinuity. This figure defines the velocities v, v', $\tilde{v}$, $\tilde{v}'$, which are taken to be constant in each sector defined by the crossing bank and layer discontinuity. Also the unit normal and tangential vectors n, t and $\tilde{n}$, $\tilde{t}$ are defined in this figure.

Figure 2. A bank crossing a layer discontinuity with first order contact. This figure serves to define dx, $d\tilde{x}$, $d\phi$ and dy.

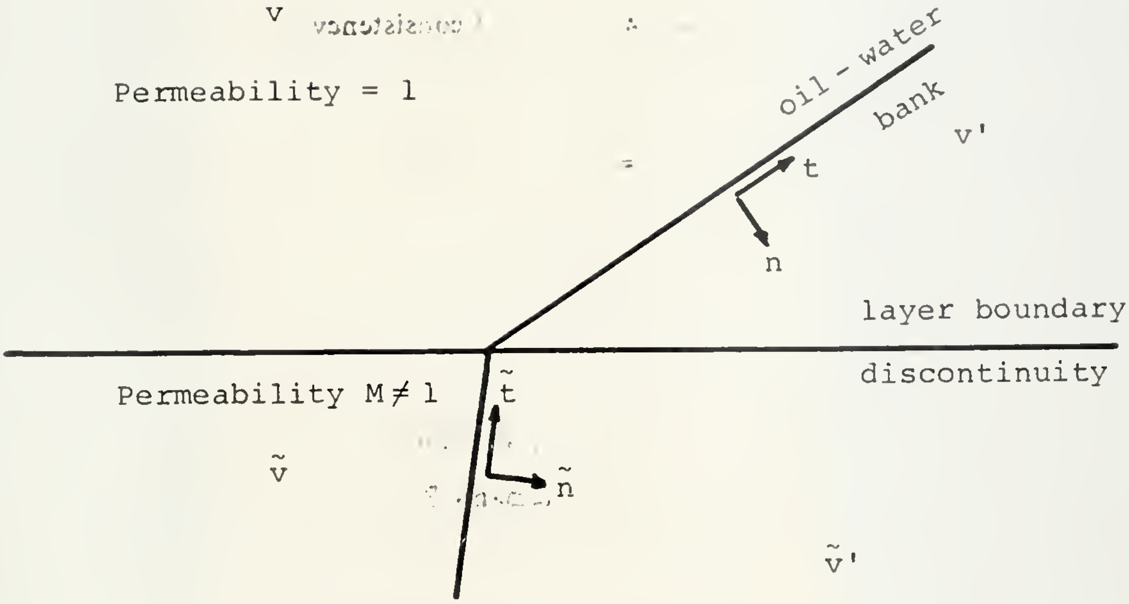

Figure 1

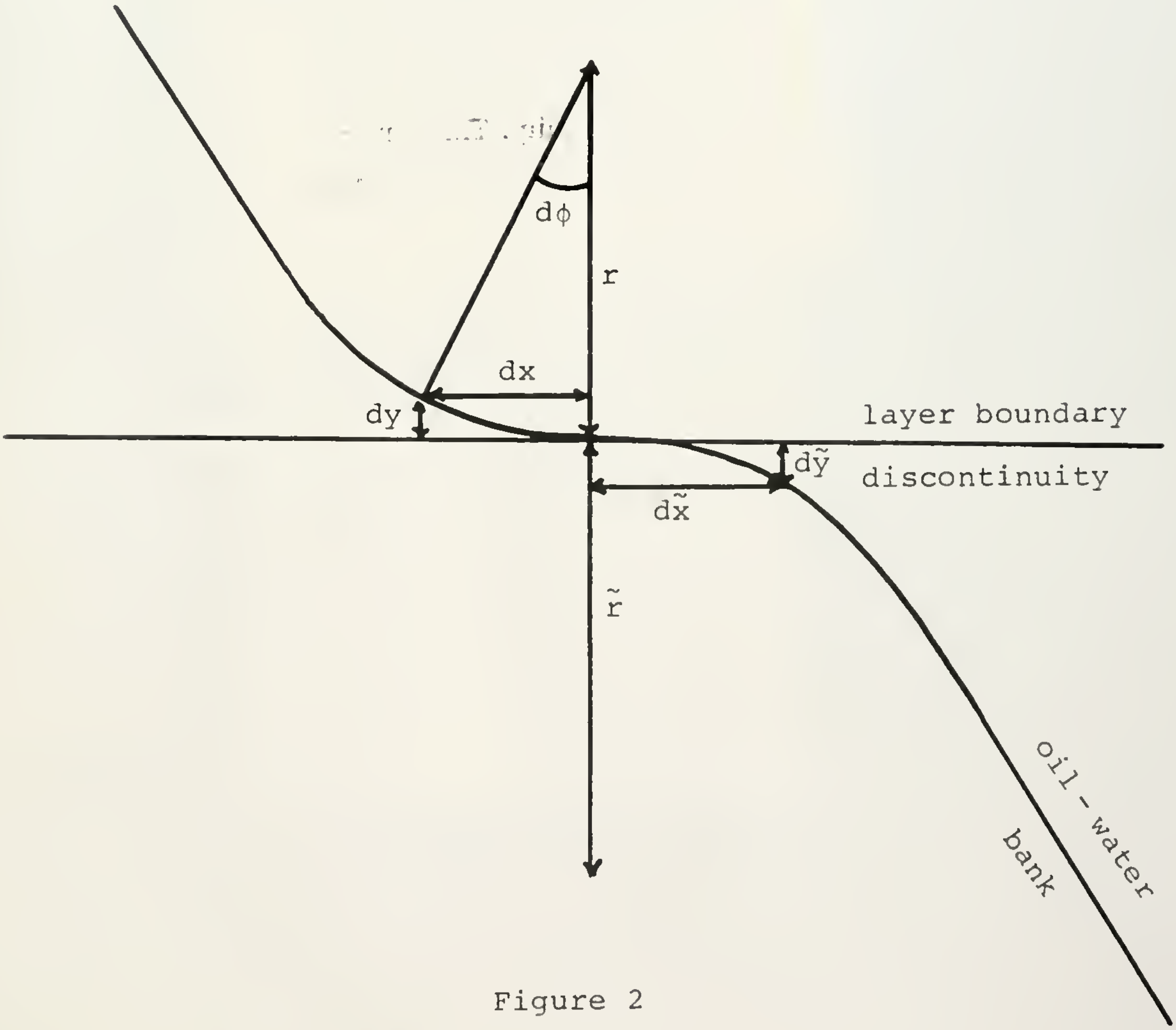

Figure 2

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